K-moduli of a family of conic bundle threefolds.

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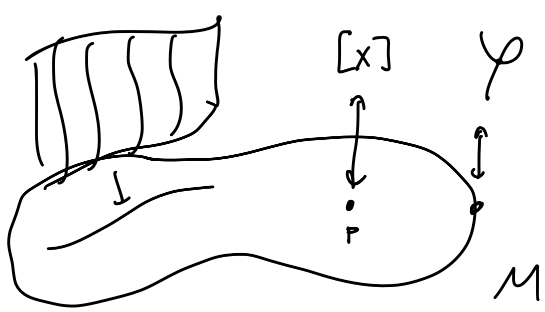
Introduction

Goal: construct a moduli space for some fixed type of variety / $\mathbb{C}$.

$\mathcal{M} = \{ [X]$: variety with fixed dim, volume, ... other properties $\}$



* give alg. structure? *

Other goal: $\mathcal{M}$ compact / proper.

· may have to allow singular objects *

Q1: which ones?

Q2: If smooth members have some nice properties, do the singular ones?

e.g. MFS structure

double cover struc.

rationality

...

For X canonically polarized

K_X ample

- know how to construct $\mathcal{M}$.

  know answer to Q1.

  (Q2: some cases but no in general)

For X Fano

$-K_X$ ample

- Only very recently do we know how to construct $\mathcal{M}$.
  - don't have as nice of an answer to Q1.

In many cases, there are several different $\mathcal{M}$'s. Q3: How are they related?

Goal for today:

construct moduli spaces of:

Fano family 2.18 : conic bundles via construction

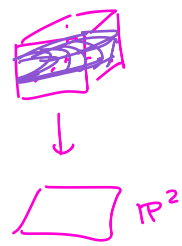
$\updownarrow$: $\{ Y \}$

Double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over
t_0, t_1 y_0, y_1, y_2

(2,2) divisor

: $\{ (X, \tfrac{1}{2} R) \}$

$t_0^2 Q_1 + 2 t_0 t_1 Q_2 + t_1^2 Q_3$



Questions:

- what objects are on the boundary?
- which structures are preserved?
  (double cover, map to $\mathbb{P}^2$, map to $\mathbb{P}^1$, conic bundle, rationality?)
- How are different moduli spaces related?

  e.g. to each Y or $(X, \tfrac{1}{2}R)$, have a <u>discriminant</u> curve $\Delta \subseteq \mathbb{P}^2$

  $\Delta = Q_1 Q_3 - Q_2^2$

  Relate to M_Δ ?

Some surprises along the way!

Main players:

K-stability and K-moduli spaces

Theorem. (Xu + ...)

For fixed $n = \dim$, $V = (-K_X - D)^n$ volume,

$\exists$ f.t. Artin stack and projective g.m.s.

$$\begin{array}{ll} \mathcal{M}^{K\text{-ss}}_{n,V} & \text{param K-ss } (X,D) \text{ log Fano} \\ \downarrow & \\ M^{K\text{-ps}}_{n,V} & \text{" K-ps "} \end{array}$$

Facts. (X,D) log Fano

- K-ss $\Rightarrow$ (X,D) is klt (Odaka)
  (X normal!)
- K-ps $\Rightarrow$ $\mathrm{Aut}(X,D)$ is reductive (ABHLX)
- K stability "like" GIT
- K stability behaves well with respect to finite covers:
  $\pi : Y \to (X,D)$ $\quad K_Y = \pi^*(K_X + D)$
  (X,D) K-ss $\Longleftrightarrow$ Y K-ss. (Zhuang)
- $\forall X$ sm Fano $\dim \le 3$, know if gen. member of each def. family is K-ss or not. (Araujo et al)

For example: known that general $\in 2.18$ is K-ss.

$\cup$ smooth Y $\quad\quad$ $\cup$ smooth R

K-moduli space of 2.18 $\quad$ K-mod of $(X, \frac{1}{2}R)$

$$\begin{tikzcd}
M^K_{2.18} \arrow[r, leftrightarrow, "="] & M^K_{c=1/2} \arrow[d, dotted, "?"] \arrow[dl, dashed] \\
M^{GIT}_\Delta & M^{GIT}_R \arrow[l, dashed]
\end{tikzcd}$$

GIT of (2,2) divisors

R smooth: Δ at worst nodal (GIT stable)

M^{GIT}_Δ: GIT of quartic discriminant.

Is gen. smooth R GIT s.s.? (yes)

(Note: R weak dP of deg 2)

Starting points: ① '23 Cheltsov, Fujita, Kishimoto, Park: all smooth Y are K-stable.

And ② ADL 19:

- There exists a "wall crossing framework" for K-ss pairs (X, cD) $\quad D \sim_{\mathbb{Q}} -rK_X$, $r \in \mathbb{Q}$

and in many settings, for $c \ll 1$, $M^K_{c\ll 1} \cong M^{GIT}_D$

$$\begin{tikzcd}[column sep=small]
 & \cdots \arrow[dl] & & M^K_{c_1+\varepsilon} \arrow[dr] \arrow[ll] & & M^K_{c_1-\varepsilon} \arrow[dl] \arrow[dr] & \\
{} & & & & M^K_{c_1} & & M^K_{c\ll 1} \cong M^{GIT}_D
\end{tikzcd}$$

and the walls occur at rational c_i.

Our case: $\underset{(2,2)}{D} \sim -r\underset{(-2,-3)}{K_X}$

But still true that for $c \ll 1$,

$\mathcal{M}^K_{c\ll 1} \cong M^{GIT}_D$, and expected that same wall crossing is true.

Goal: describe all GIT ss (2,2) divisors R and determine if any have to be "replaced" (wall) as c increases to $1/2$.

$\rightsquigarrow$ compute description of $M^K_{c=1/2} = \mathcal{M}^k_{2.18}$

<u>Comment</u>: some partial GIT results for (2,2): Parameswaran, Ray

<u>Theorem</u>. (DJKHQ)

For a (2,2) surface $R \subseteq \mathbb{P}^1 \times \mathbb{P}^2$, R is GIT stable if and only if:

Any singularity $p \in R$ is an A_n singularity, $1 \le n \le 7$, and satisfies:

(a) if $p \in R$ is an A_1 singularity, then p is the only singular point contained in the fiber of $\pi_2 : R \to \mathbb{P}^2$ over $\pi_2(p)$

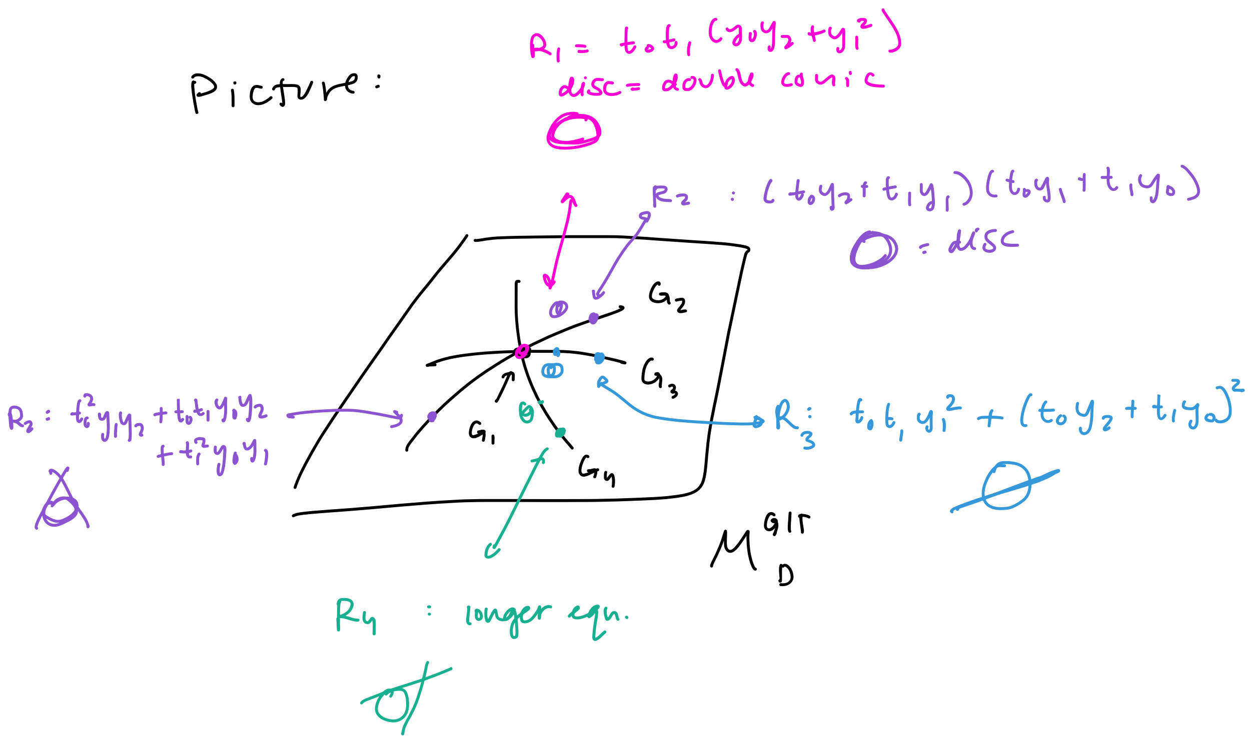
(b) if $p \in R$ is an A_n sing, $n \geq 2$, then the fiber over $\pi_2(p)$ is finite and fiber of π_1 is reduced.

Theorem. (DJKHQ)

A GIT stable (2,2) surface is K-stable for all $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$.

Theorem (DJKHQ). The GIT (semi)/polystable locus is explicitly described.

Picture:



and:

- Map to M^{GIT}_Δ is defined away from R_3.
- $\forall R \neq R_3$, (X, cR) is K-ps $\forall c \in (0, \frac{1}{2}]$.

Proof ideas:

- $R = (t_0^2Q_1 + 2t_0t_1Q_2 + t_1^2Q_3) \Rightarrow \Delta = Q_1Q_3 - Q_2^2$

- Δ GIT s.s. $\Rightarrow$ R is GIT s.s. (converse false)
- use that $\underset{(X, cR)}{K\text{-ss}} \Rightarrow \underset{R}{\text{GIT ss.}}$
- check K-ss directly with theory of flags.

A surprise!

Theorem: $c_0 :=$ smallest sol. to $10c^3 - 34c^2 + 35c - 10 = 0.$

Then, (X, cR) K-ss $\Longleftrightarrow$ $c \leq c_0$.

"k.s.t"

Corollary. There exist irreducible divisors on smooth Fano threefolds s.t. kst (X, R) is irrational.

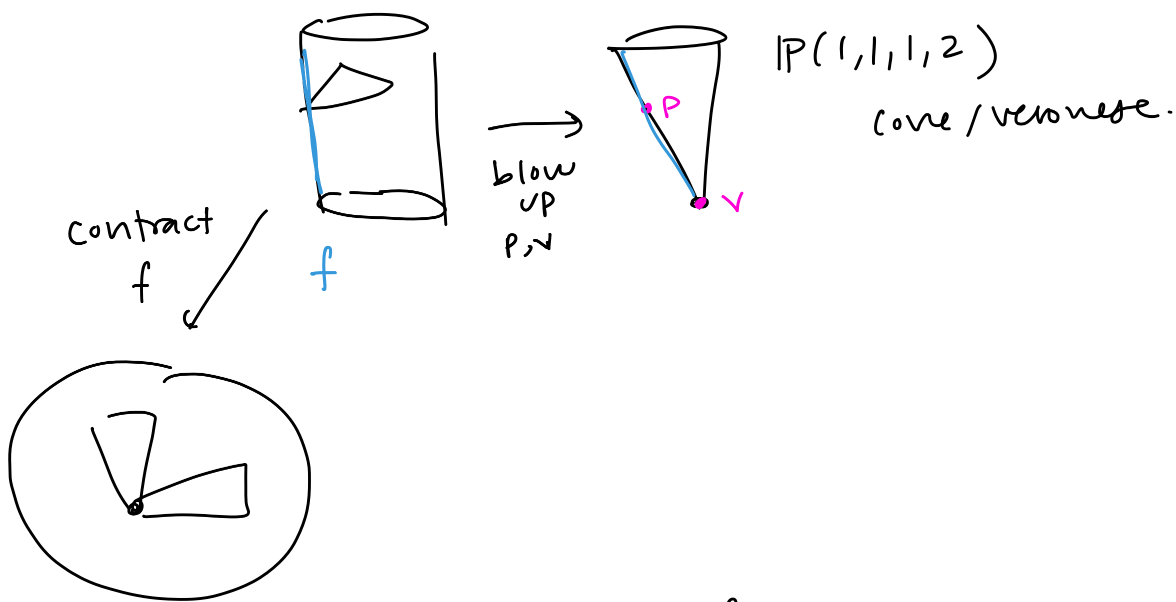
[First such example!]

Back to moduli:

If we "replace" R_3, should resolve map to GIT of Δ and give $\mathcal{M}^K_{2.18}$.

How to replace?

Let X_3 be constructed as follows.



X_3 : non $\mathbb{Q}$-factorial

Theorem. X_3 is canonical Gorenstein (one ODP) and admits a smoothing to $\mathbb{P}^1 \times \mathbb{P}^2$.

If

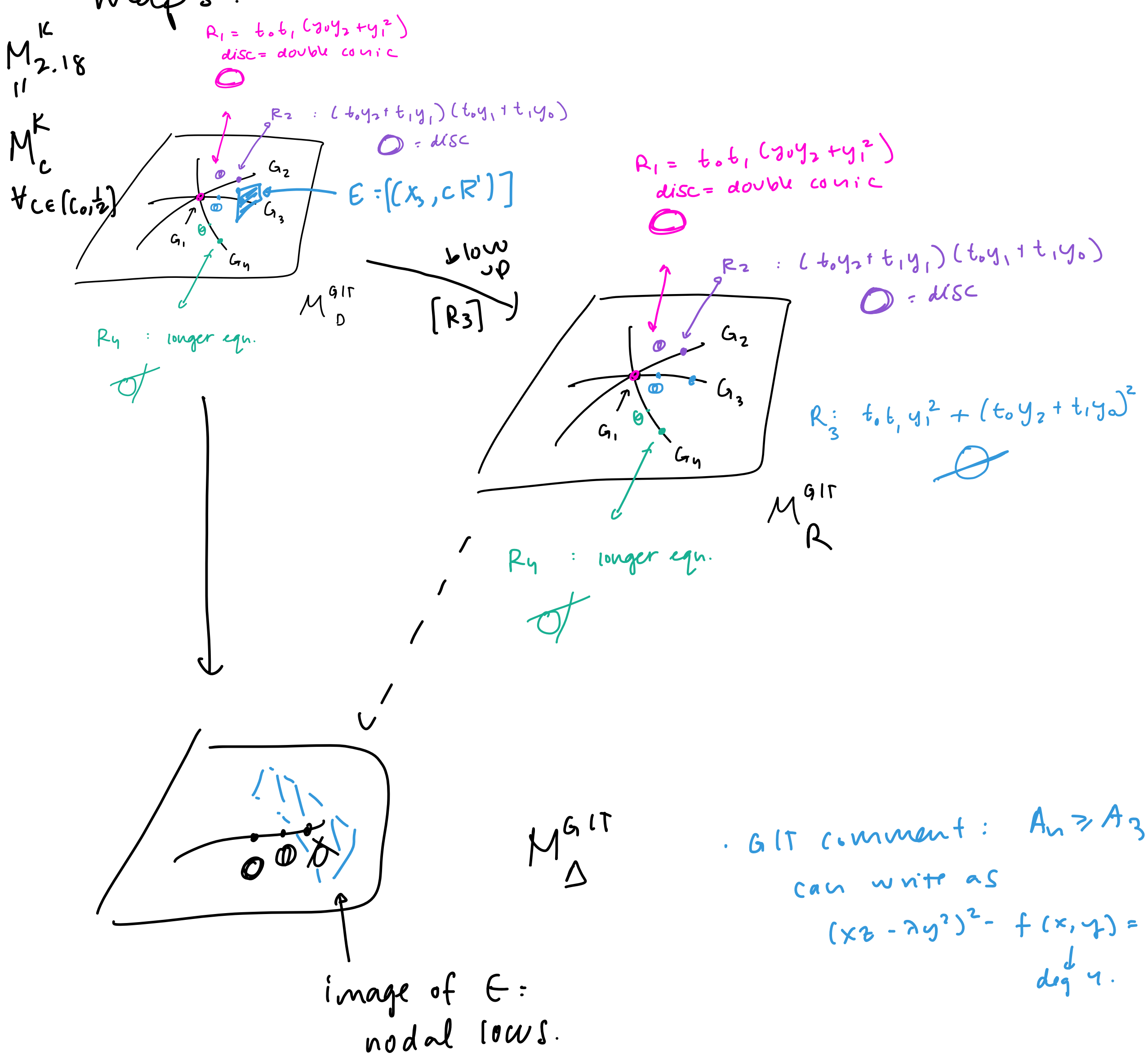
$$(\mathbb{P}^1 \times \mathbb{P}^2, cR_t) \overset{K\text{-ss}}{\rightsquigarrow} (\mathbb{P}^1 \times \mathbb{P}^2, cR_3)$$

and $c \geq c_0$

Then this can be birationally modified to $(\mathbb{P}^1 \times \mathbb{P}^2, cR_t) \rightsquigarrow (X_3, cR')$ K-ps limit.

And, for every R' with singularities as in $\mathcal{M}^{GIT}_R - [R_3]$, (X_3, cR') is K-ss for $c \in [0, 1/2]$.

So: we have explicitly resolved several maps.

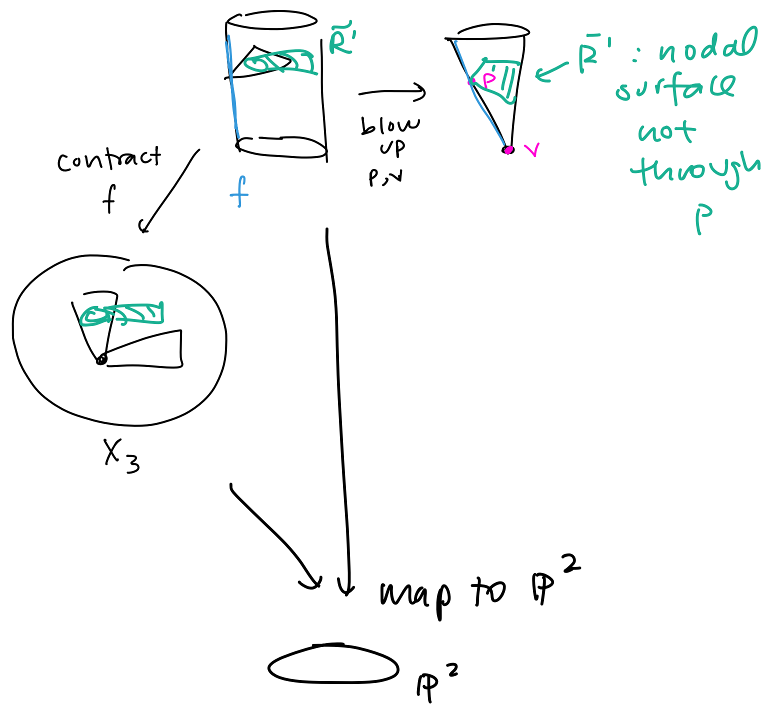


· GIT comment: $A_n \geq A_3$, can write as $(xz - \lambda y^2)^2 - f(x,y) = 0$, f deg 4.

So: Q1: what is on the boundary? we know.

Q2: what structures specialize?

$Y_3 \xrightarrow[\text{over } R']{2:1} X_3$



still "conic bundle" just not flat (one fiber is a surface)

preserved!

* no map to $\mathbb{P}^1$ *

de Fernex - Fri: rationality specializes in klt of dim ≤ 3

so X_3, Y_3 are rational.

- Gives complete description of K-moduli of 2.18, and insight? examples? to more general phenomena.